Algorithms for Exact Real Arithmetic using Möbius Transformations

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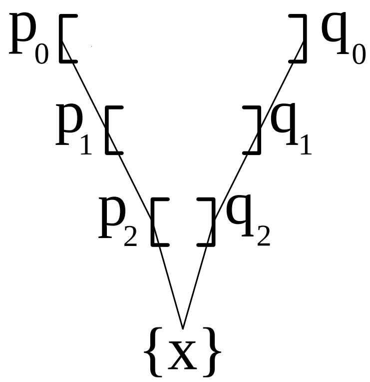
Real Numbers $\mathbb{R}$

Sequences of nested closed intervals



$$[p_0, q_0] \supseteq [p_1, q_1] \supseteq [p_2, q_2] \supseteq \cdots$$

$$|p_n - q_n| \to 0 \text{ as } n \to \infty$$

$$\bigcap_{n \in \mathbb{N}} [p_n, q_n] = \{x\}$$

Restrict end points to countable subset $\mathbb{F} \subset \mathbb{R}$

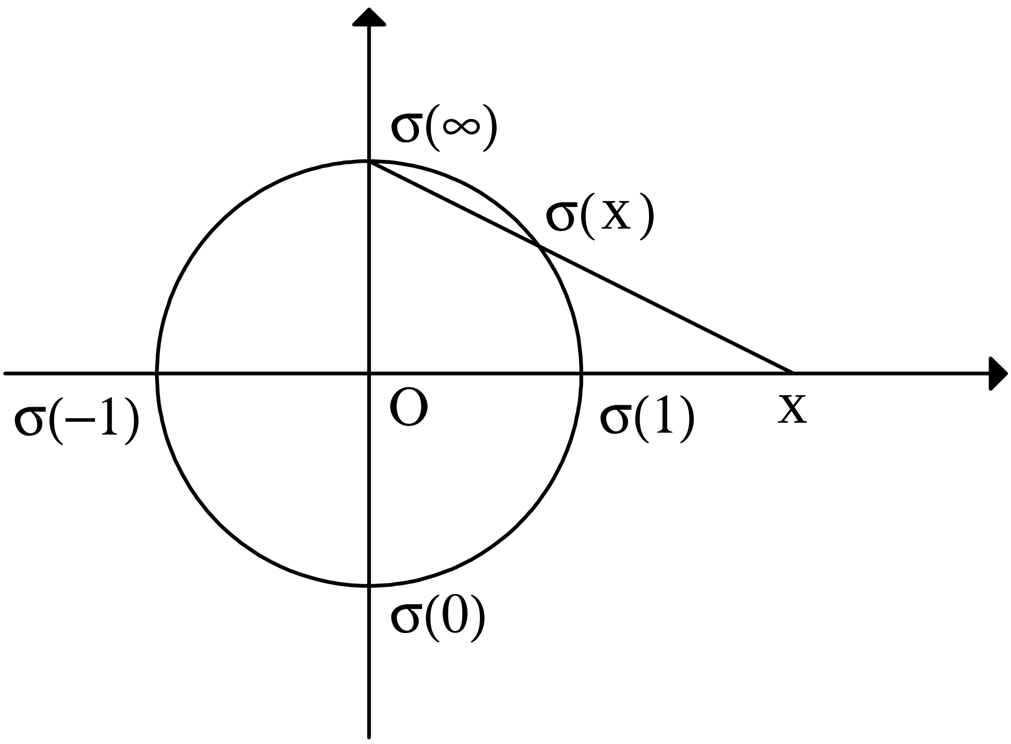
(a) Rational numbers $\mathbb{Q}$

(b) 2-adic numbers $\left\{ \frac{n}{2^m} \middle| n, m \in \mathbb{Z} \right\}$

(c) Quadratic field $\mathbb{Q}\left(\sqrt{5}\right) = \left\{ p + q\sqrt{5} \middle| p, q \in \mathbb{Q} \right\}$

Add Infinity

- All real numbers have reciprocals except 0.
- But we cannot necessarily detect 0.
- So we have to include 0^{-1} denoted by ∞.
- This is known as the one-point compactification of the real line denoted $\mathbb{R}^{\infty}$.



$$\sigma(x) = \left(\frac{2x}{x^2+1}\right) + \left(\frac{x^2-1}{x^2+1}\right) i$$

Add Bottom

- However, including ∞ leads to other difficulties.

- What are we to make of $0 \times \infty$?

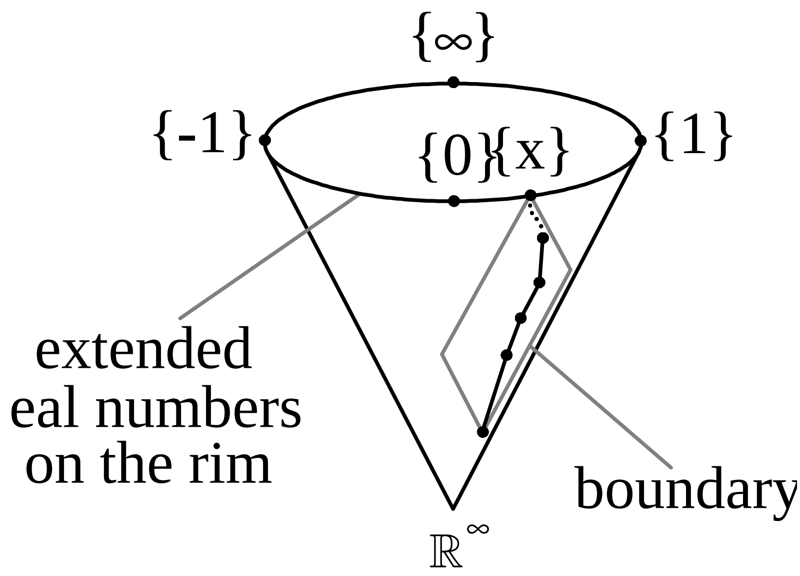
- We know that for any real number x
$$0 \times x = 0$$
and for any non-zero real number x
$$x \times x^{-1} = 1.$$

- The only sensible answer is to introduce the concept of an **"undefined number"** or **"not a number"** denoted by **NaN** in the floating point community.

- **Domain theoretically**, this object is of course bottom denoted $\perp$.

Continuous Domain

$$(\mathbb{IR}^\infty, \supseteq)$$



- $\mathbb{IR}^\infty$ is the **set of closed intervals** in $\mathbb{R} \cup \{\infty\}$ including intervals through ∞.

- **For example**:
  $[1, -1]$ denotes the set
  $\{x \in \mathbb{R} \mid x \leq -1\} \cup \{x \in \mathbb{R} \mid x \geq 1\} \cup \{\infty\}$.

Vectors

- Use **Vectors** to represent **extended rational numbers**

$$a, b \in \mathbb{Z} - \{0\}$$

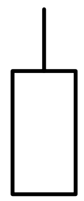
$$\Phi \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \perp$$

$$\Phi \begin{pmatrix} a \\ 0 \end{pmatrix} = \infty$$

$$\Phi \begin{pmatrix} a \\ b \end{pmatrix} = \frac{a}{b}$$

- **Drop** Φ for convenience

- **Picture** representation



- **Note scaling invariance**

Decimal Representation

- **End points**

$$\left\{ \frac{n}{10^m} \middle| n, m \in \mathbb{Z} \right\}$$

- **Base interval**

$$[0, 1]$$

- **Digit set**

$$d \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

- **Digit map**

$$f_d(x) = \frac{x + d}{10}$$

- **Example**

$$0\,.\,\boxed{3}\boxed{1}\boxed{7}\boxed{4}\;\cdots$$

$$\Downarrow$$

$$\begin{aligned}
f_3([0,1]) &= \left[\tfrac{3}{10}, \tfrac{4}{10}\right] \\
f_3(f_1([0,1])) &= \left[\tfrac{31}{100}, \tfrac{32}{100}\right] \\
f_3(f_1(f_7([0,1]))) &= \left[\tfrac{317}{1000}, \tfrac{318}{1000}\right] \\
f_3(f_1(f_7(f_4([0,1])))) &= \left[\tfrac{3174}{10000}, \tfrac{3175}{10000}\right]
\end{aligned}$$

Efficient algorithms for $+$ and $-$ of finite representations

Redundant Binary Representation

- **End points**

$$\left\{ \frac{n}{2^m} \middle| n, m \in \mathbb{Z} \right\}$$

- **Base interval**

$$[-1, 1]$$

- **Digit set**

$$d \in \{-1, 0, 1\}$$

- **Digit map**

$$f_d(x) = \frac{x + d}{2}$$

- **Example**

$$0\,.\,\boxed{1}\boxed{-1}\boxed{0}\boxed{1}\cdots$$

$$\Downarrow$$

$$\begin{aligned} f_1([-1,1]) &= [0,1] \\ f_1(f_{-1}([-1,1])) &= \left[0, \tfrac{1}{2}\right] \\ f_1(f_{-1}(f_0([-1,1]))) &= \left[\tfrac{1}{8}, \tfrac{3}{8}\right] \\ f_1(f_{-1}(f_0(f_1([-1,1])))) &= \left[\tfrac{1}{4}, \tfrac{3}{8}\right] \end{aligned}$$

$\boxed{\textbf{Efficient}\text{ algorithms for } + \text{ and } - \text{ of finite representations}}$

Continued Fraction Representation

- **End points**

$$\mathbb{Q}$$

- **Base interval**

$$[1, \infty]$$

- **Digit set**

$$d \in \{1, 2, 3, 4, 5, \ldots\}$$

- **Digit map**

$$f_d(x) = d + \frac{1}{x}$$

- **Example**

$$\sqrt{2} = \boxed{1} + \cfrac{1}{\boxed{2} + \cfrac{1}{\boxed{2} + \cfrac{1}{\boxed{2} + \cdots}}}$$

$$\Downarrow$$

$$\begin{aligned}
f_1([1, \infty]) &= [1, 2] \\
f_1(f_2([1, \infty])) &= \left[\tfrac{4}{3}, \tfrac{3}{2}\right] = [1.333\ldots, 1.5] \\
f_1(f_2(f_2([1, \infty]))) &= \left[\tfrac{7}{5}, \tfrac{10}{7}\right] = [1.4, 1.428\ldots] \\
f_1(f_2(f_2(f_2([1, \infty])))) &= \left[\tfrac{24}{17}, \tfrac{17}{12}\right] = [1.411\ldots, 1.416\ldots]
\end{aligned}$$

Elegant algorithms for *transcendental functions*

Matrices

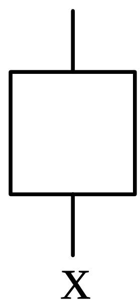
- Use **Matrices** to represent **Möbius transformations**

$$a, b, c, d \in \mathbb{Z} \text{ and } x \in \mathbb{R}$$

$$\Psi \begin{pmatrix} a & c \\ b & d \end{pmatrix} (x) = \frac{ax + c}{bx + d}$$

- **Drop** Ψ for convenience

- **Picture** representation



- **Note scaling invariance**

Properties

- **Composition** of **Möbius transformations** is equivalent to **product of matrices**

$$M\left(N\left(x\right)\right) = \left(M \bullet N\right)\left(x\right)$$

- **Application** of **Möbius transformations** to **extended rational numbers** is equivalent to **product of matrices and vectors**

$$M\left(V\right) = M \bullet V$$

- **What do singular matrices represent?**

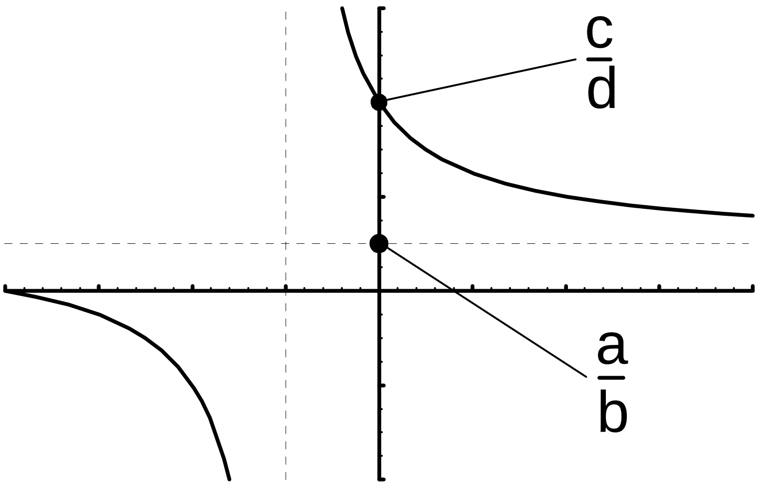
$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(x) = 0 \times x = \begin{cases} 0 & \text{if } x \neq \infty \\ \perp & \text{if } x = \infty \end{cases}$$

because

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix} \equiv \begin{cases} 0 & \text{if } b \neq 0 \\ \perp & \text{if } b = 0 \end{cases}$$

Special Base Interval

$$[0, \infty]$$



$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}([0,\infty]) = \begin{cases} \left[\dfrac{a}{b}, \dfrac{c}{d}\right] & \text{if } \det\begin{pmatrix} a & c \\ b & d \end{pmatrix} < 0 \\ \left[\dfrac{c}{d}, \dfrac{a}{b}\right] & \text{if } \det\begin{pmatrix} a & c \\ b & d \end{pmatrix} > 0 \end{cases}$$

The Refinement Property

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}([0,\infty]) \subseteq [0,\infty]$$

iff

$$\begin{pmatrix} a,b,c,d \geq 0 \\ \text{or} \\ a,b,c,d \leq 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Unsigned General Normal Product

- **End points**

$$\mathbb{Q}$$

- **Base interval**

$$[0, \infty]$$

- **Digit set**

$$M \in \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{N} \right\}$$

- **Digit map**

$$f_M(x) = M(x)$$

- **Example**

$$e = \begin{array}{|c|c|c|c|} \hline M_0 & M_1 & M_2 & M_3 \\ \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} & \begin{pmatrix} 6 & 5 \\ 5 & 4 \end{pmatrix} & \begin{pmatrix} 6 & 5 \\ 5 & 4 \end{pmatrix} \\ \hline \end{array} \cdots$$

$$\Downarrow$$

$$\begin{aligned} M_0([0,\infty]) &= [1,\infty] \\ M_0(M_1([0,\infty])) &= [2,3] \\ M_0(M_1(M_2([0,\infty]))) &= \left[\tfrac{5}{2}, \tfrac{11}{4}\right] = [2.5, 2, 75] \\ M_0(M_1(M_2(M_3([0,\infty])))) &= \left[\tfrac{8}{3}, \tfrac{49}{18}\right] = [2.666\ldots, 2, 722\ldots] \end{aligned}$$

Comparisons

- **Möbius transformation**

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}(x) = \frac{ax+c}{bx+d}$$

- **Decimal representation**

$$f_d(x) = \frac{x+d}{10} = \begin{pmatrix} 1 & d \\ 0 & 10 \end{pmatrix}(x)$$

- **Redunandant binary representation**

$$f_d(x) = \frac{x+d}{2} = \begin{pmatrix} 1 & d \\ 0 & 2 \end{pmatrix}(x)$$

- **Continued fraction representation**

$$f_d(x) = d + \frac{1}{x} = \begin{pmatrix} d & 1 \\ 1 & 0 \end{pmatrix}(x)$$

Signed General Normal Product

- **Sign set**

$$M \in \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z} \right\}$$

- **Sign map**

$$f_M(x) = M(x)$$

- **Unsigned general normal product**

$$x$$

- **Example**

M_0	M_1	M_2	M_3	
$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$	$\cdots$

$$\Downarrow$$

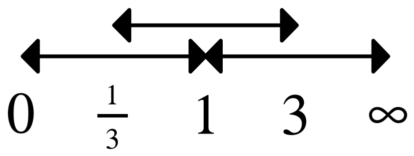
$$\begin{aligned} M_0([0,\infty]) &= [-1,1] \\ M_0(M_1([0,\infty])) &= [0,1] \\ M_0(M_1(M_2([0,\infty]))) &= \left[\tfrac{1}{4}, \tfrac{3}{4}\right] \\ M_0(M_1(M_2(M_3([0,\infty])))) &= \left[\tfrac{1}{4}, \tfrac{1}{2}\right] \end{aligned}$$

Unsigned Exact Floating Point

3 digit matrices

$$D_0 \stackrel{\text{def}}{=} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$D_{-1} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$



$$D_1 \stackrel{\text{def}}{=} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

Conjugate to the redundant binary

$$\begin{tikzcd}
{[-1,1]} \arrow[r, "\frac{d+x}{2}"] & {[-1,1]} \\
{[0,\infty]} \arrow[u, "\frac{x-1}{x+1}"'] \arrow[r, "D_d"'] & {[0,\infty]} \arrow[u, "\frac{x-1}{x+1}"']
\end{tikzcd}$$

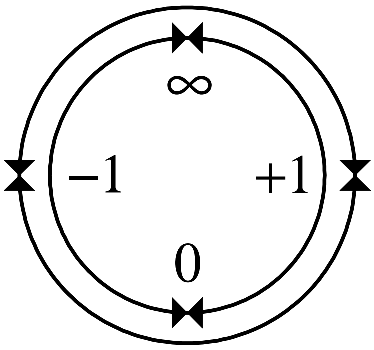
$$D_d = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & d \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3+d & 1+d \\ 1-d & 3-d \end{pmatrix}$$

Signed Exact Floating Point

4 sign matrices

$$S_\infty \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$S_- \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



$$S_+ \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_0 \stackrel{\text{def}}{=} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{aligned} S_+([0,\infty]) &= [0,\infty] \\ S_\infty([0,\infty]) &= [1,-1] \\ S_-([0,\infty]) &= [\infty,0] \\ S_0([0,\infty]) &= [-1,1] \end{aligned}$$

Form a cyclic group of order 4

$$\begin{aligned} S_\infty^1 &= S_\infty = \text{ rotation by } \tfrac{\pi}{2} \\ S_\infty^2 &= S_- \\ S_\infty^3 &= S_0 \\ S_\infty^4 &= S_+ \end{aligned}$$

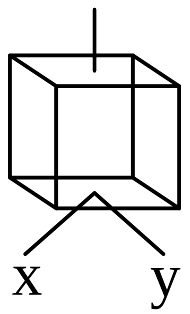
Tensors

- Use **Tensors** to represent
  2-D Möbius transformations

$$a, b, c, d, e, f, g, h \in \mathbb{Z} \text{ and } x, y \in \mathbb{R}$$

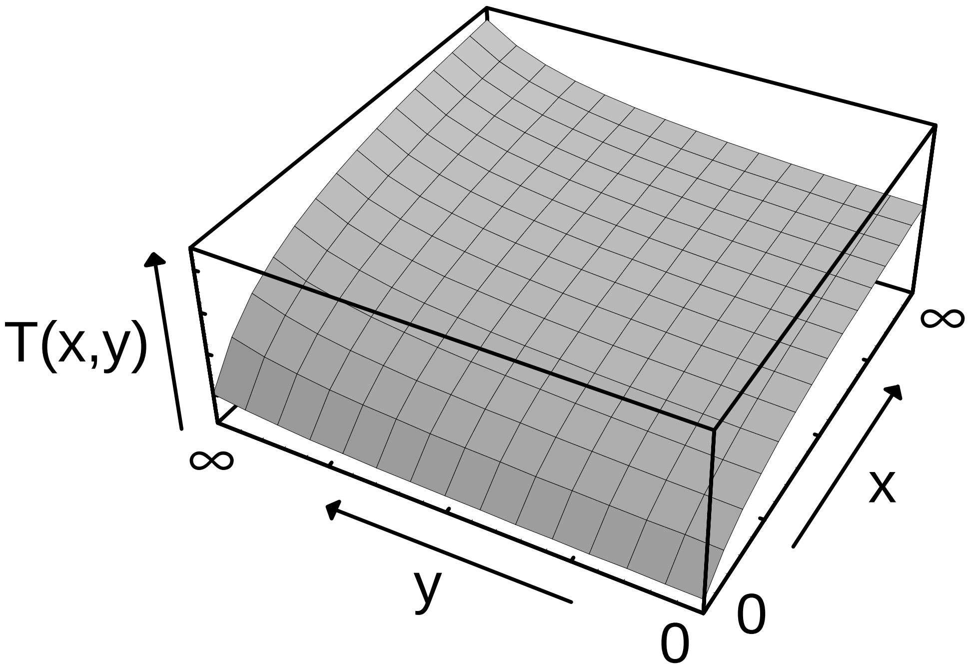
$$\Upsilon \begin{pmatrix} a & c & e & g \\ b & d & f & h \end{pmatrix} (x, y) = \frac{axy + cx + ey + g}{bxy + dx + fy + h}$$

- Picture representation



Information in a tensor

$$\begin{pmatrix} a & c & e & g \\ b & d & f & h \end{pmatrix} ([0,\infty], [0,\infty])$$



$$\begin{aligned} &\begin{pmatrix} a & c & e & g \\ b & d & f & h \end{pmatrix} ([0,\infty], [0,\infty]) \\ = \; &\begin{pmatrix} a & c \\ b & d \end{pmatrix} ([0,\infty]) \cup \begin{pmatrix} e & g \\ f & h \end{pmatrix} ([0,\infty]) \cup \\ &\begin{pmatrix} a & e \\ b & f \end{pmatrix} ([0,\infty]) \cup \begin{pmatrix} c & g \\ d & h \end{pmatrix} ([0,\infty]) \end{aligned}$$

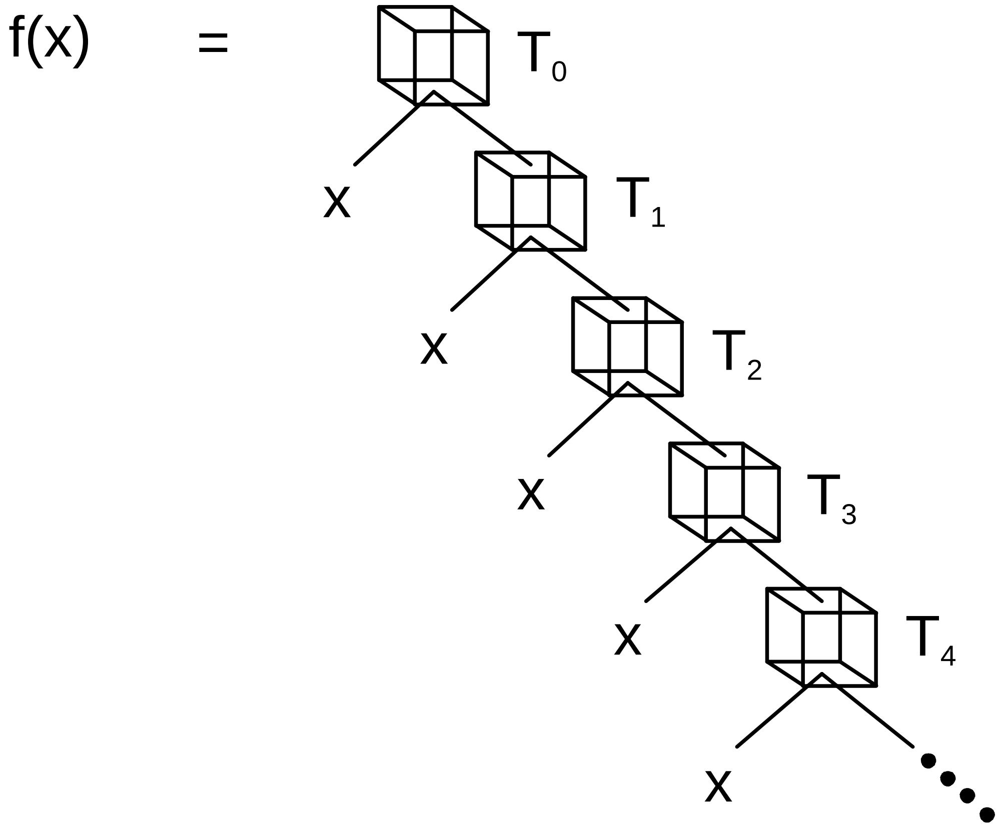
Basic Arithmetic Operations

$$\begin{pmatrix} a & c & e & g \\ b & d & f & h \end{pmatrix}(x,y) = \frac{axy + cx + ey + g}{bxy + dx + fy + h}$$

$$\begin{aligned}
T_{+}(x,y) &= \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}(x,y) = x + y \\
T_{-}(x,y) &= \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}(x,y) = x - y \\
T_{\times}(x,y) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}(x,y) = x \times y \\
T_{\div}(x,y) &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}(x,y) = x \div y
\end{aligned}$$

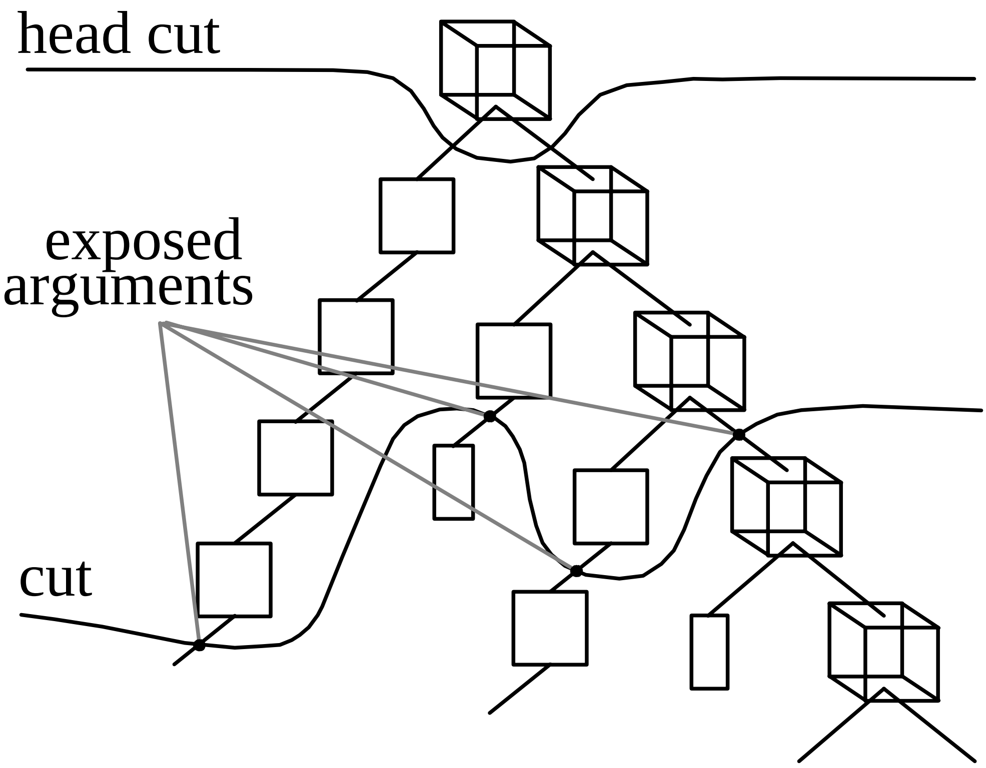
Transcendental Functions

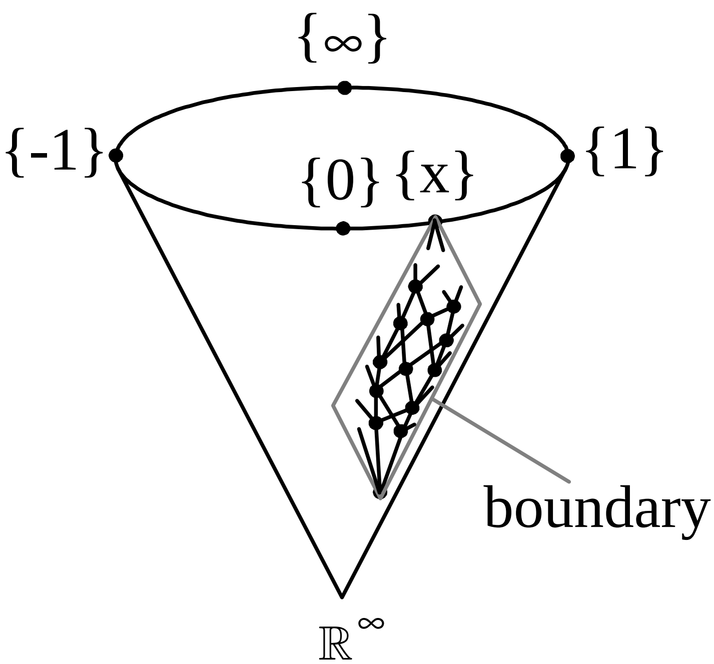
Transcendental functions can be constructed using functions of the form



This has been done for sin, cos, tan, arctan, exp, ln, tanh, arcsinh, arctanh and pow.

Expression Trees



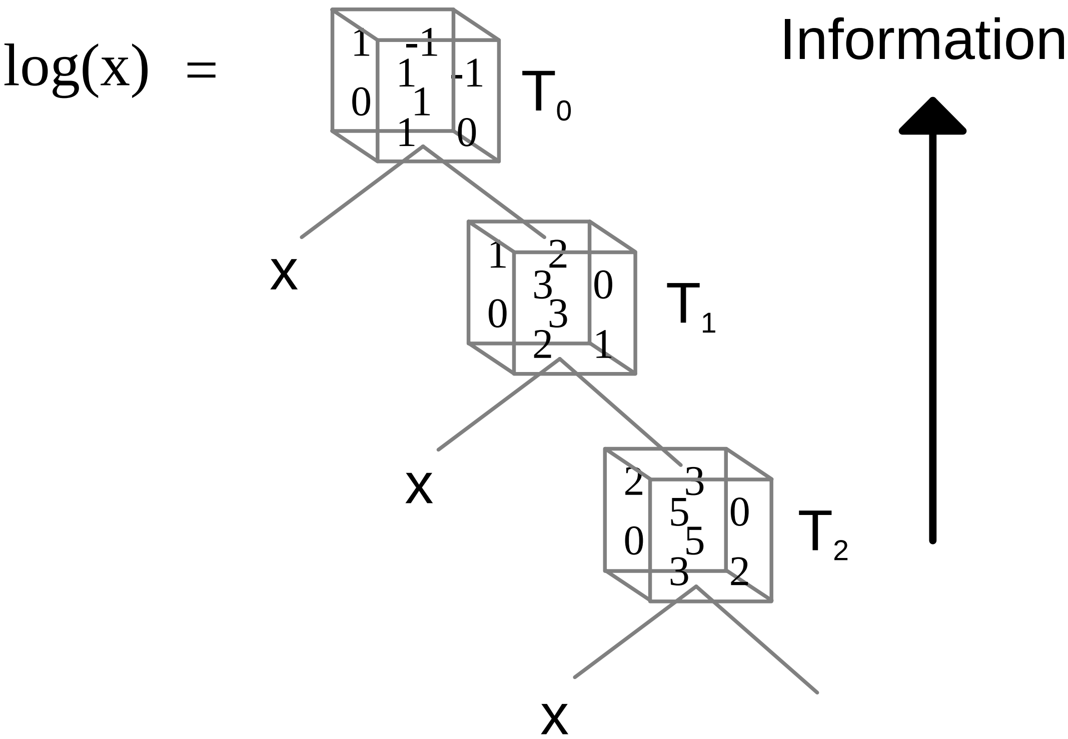


Logarithm

- $\mathbb{R}^\infty \Longrightarrow \left[\frac{1}{2}, 2\right]$

$$\log(x) = \log\left(\frac{x}{2}\right) + \log(2)$$
$$\log(x) = \log(2x) - \log(2)$$

- $\left[\frac{1}{2}, 2\right]$



with

$$T_0 = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$T_{n\geq 1} = \begin{pmatrix} n & 2n+1 & n+1 & 0 \\ 0 & n+1 & 2n+1 & n \end{pmatrix}$$

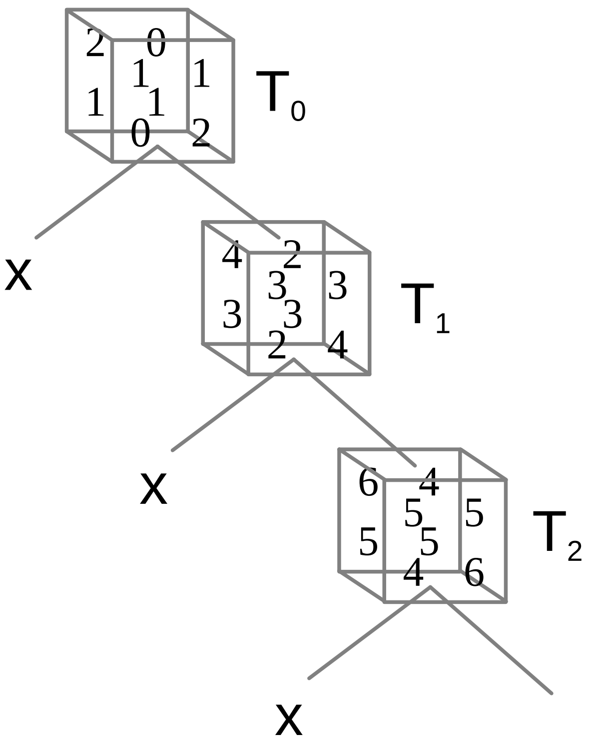
Exponential

- $\mathbb{R}^\infty \Longrightarrow [-1,1]$

$$\mathsf{exp}\,(x) = \left(\mathsf{exp}\left(\frac{x}{2}\right)\right)^2$$

- $[-1,1]$

$\mathrm{xp}(S_0(x)) =$



with

$$T_n = \begin{pmatrix} 2n+2 & 2n+1 & 2n & 2n+1 \\ 2n+1 & 2n & 2n+1 & 2n+2 \end{pmatrix}$$

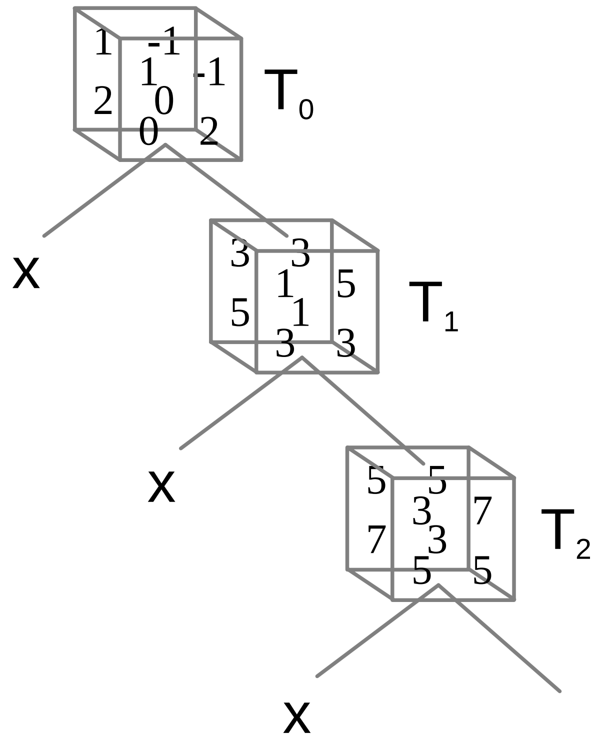
Tangent

- $\mathbb{R}^{\infty} \Longrightarrow [-1,1]$

$$\mathsf{tan}\left(x\right) = \frac{2\mathsf{tan}\left(\frac{x}{2}\right)}{1-\left(\mathsf{tan}\left(\frac{x}{2}\right)\right)^{2}}$$

- $[-1,1]$

an(S_0(x)) =



with

$$T_0 = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

$$T_{n\geq 1} = \begin{pmatrix} 2n+1 & 2n-1 & 2n+1 & 2n+3 \\ 2n+3 & 2n+1 & 2n-1 & 2n+1 \end{pmatrix}$$

Arctangent

- $\mathbb{R}^\infty \Longrightarrow [-1,1]$

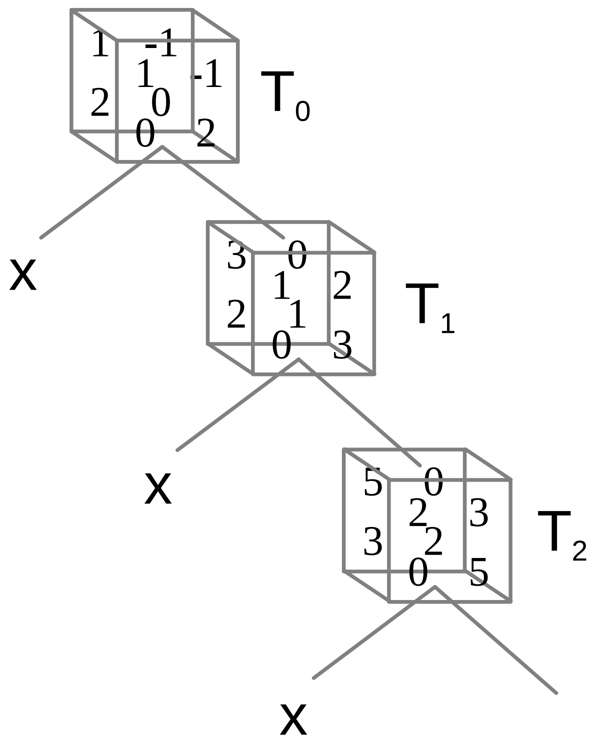
$$\mathsf{arctan}\left(S_+\left(x\right)\right) = \mathsf{arctan}\left(S_0\left(x\right)\right) + \frac{\pi}{4}$$

$$\mathsf{arctan}\left(S_\infty\left(x\right)\right) = \mathsf{arctan}\left(S_0\left(x\right)\right) + \frac{\pi}{2}$$

$$\mathsf{arctan}\left(S_-\left(x\right)\right) = \mathsf{arctan}\left(S_0\left(x\right)\right) + \frac{3\pi}{4}$$

- $[-1,1]$

rctan(S_0(x)) =

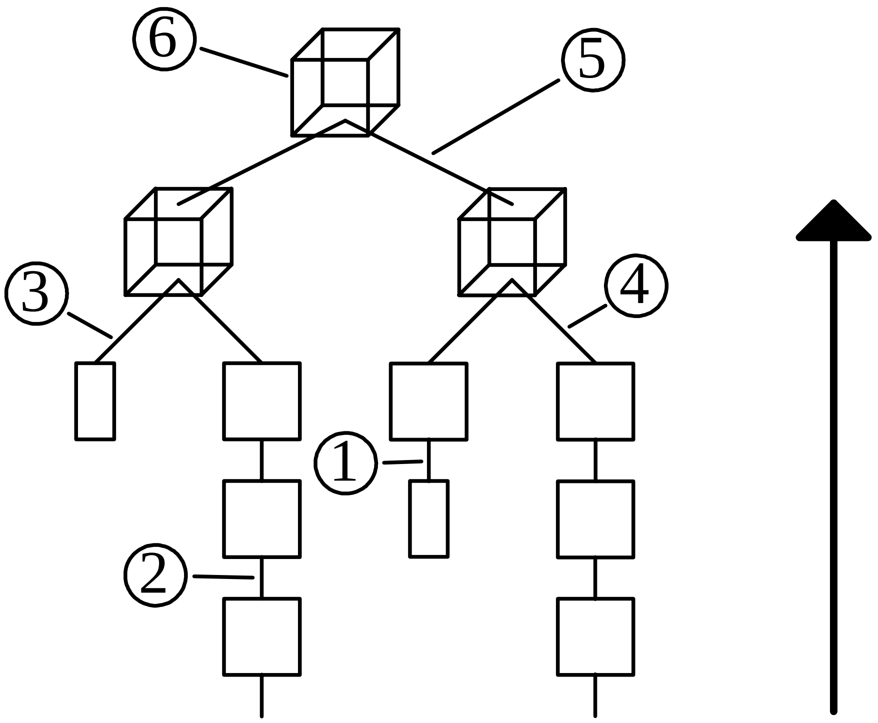


with

$$T_0 = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

$$T_{n\geq 1} = \begin{pmatrix} 2n+1 & n & 0 & n+1 \\ n+1 & 0 & n & 2n+1 \end{pmatrix}$$

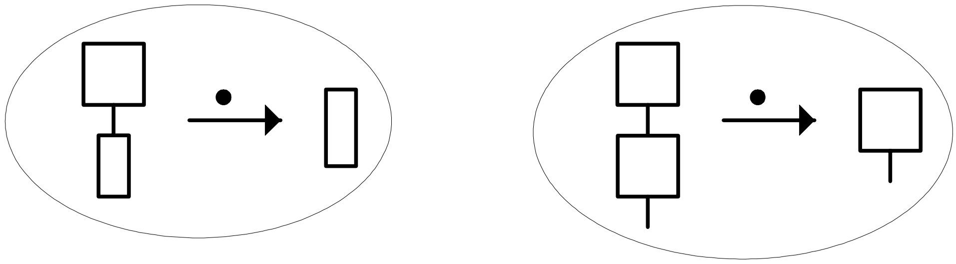
Converting Expression Tree to Exact Floating Point



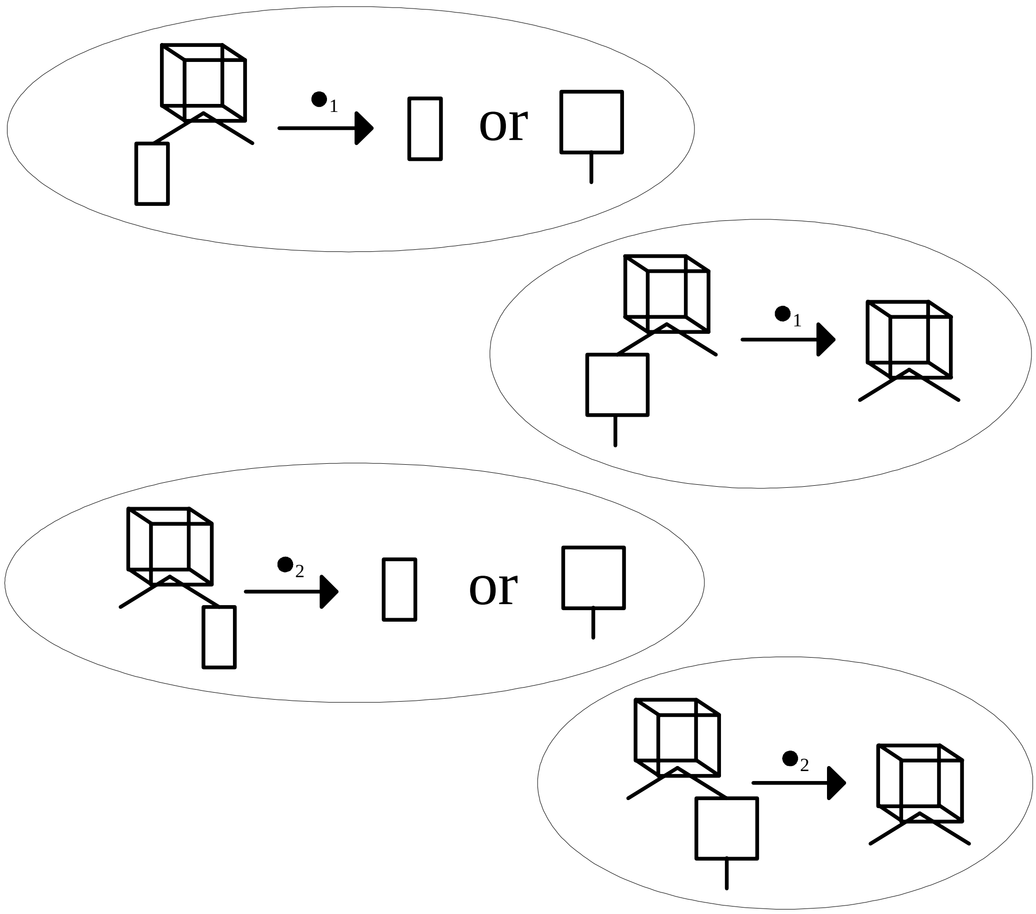
- **Reduction Rules**
  1. **Absorb** vector into matrix to give vector
  2. **Absorb** matrix into matrix to give matrix
  3. **Absorb** vector into tensor to give vector or matrix
  4. **Absorb** matrix into tensor to give tensor
  5. **Exchange** digit matrices between tensors
  6. **Emit** exact floating point from root node

Absorption Rules

- **Absorption into matrices**

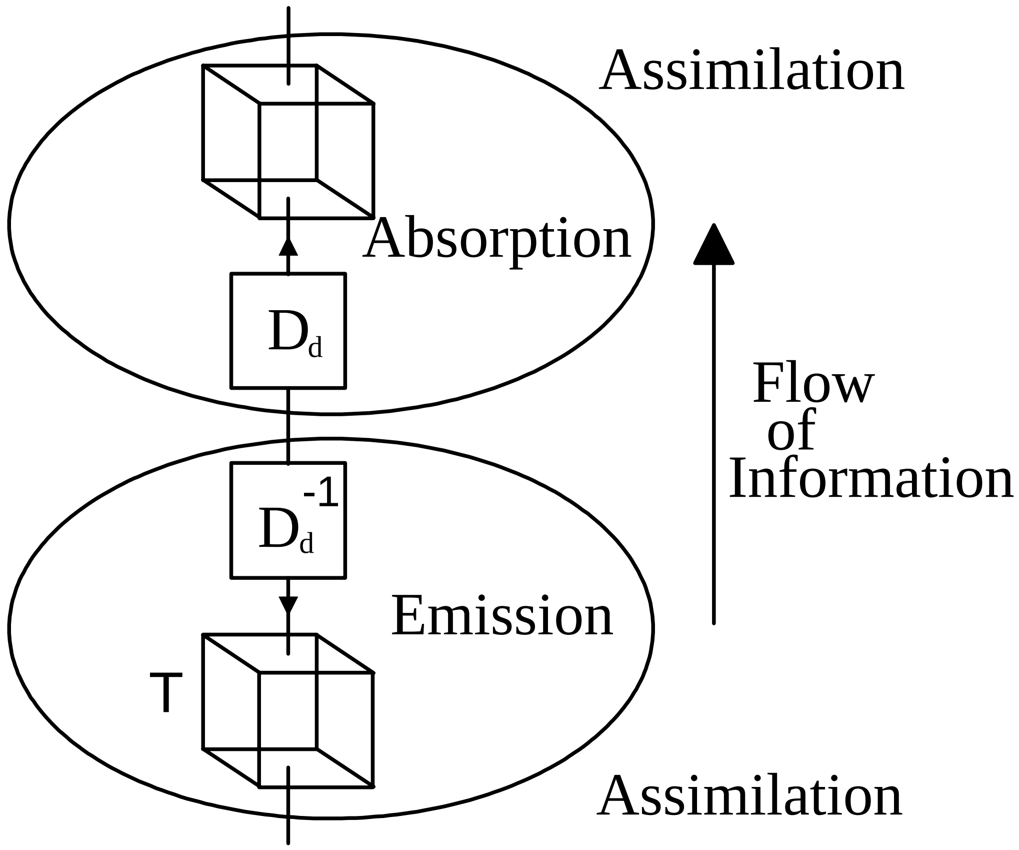


- **Absorption into tensors**



Exchange Rule

- **Exchange digit matrices between tensors**

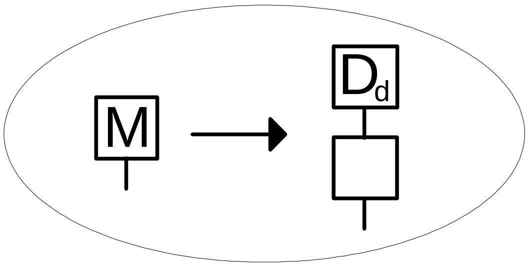


provided

$$D_d\left([0,\infty]\right) \supseteq T\left([0,\infty]\right)$$

Emission Rules

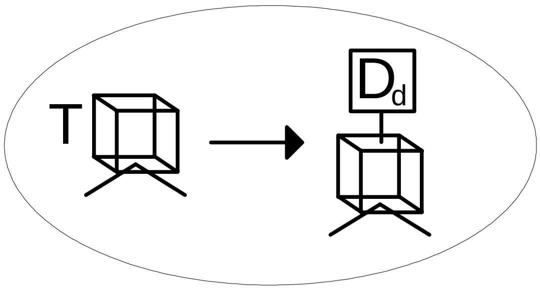
- **Emit digit matrix (or sign matrix) from matrix**



provided

$$D_d\left([0,\infty]\right) \supseteq M\left([0,\infty]\right)$$

- **Emit digit matrix (or sign matrix) from tensor**

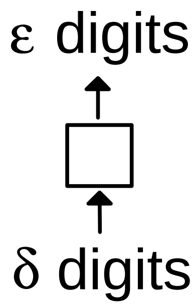


provided

$$D_d\left([0,\infty]\right) \supseteq T\left([0,\infty]\right)$$

Matrix Information Flow Analysis

- **Arbitrary matrix** in expression tree



$$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

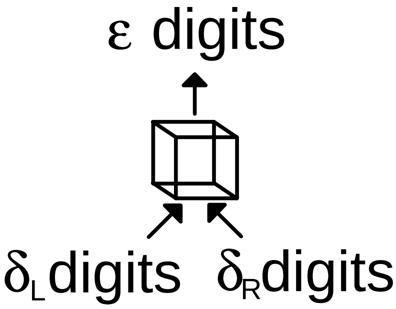
- Want to **emit** ϵ digits
- However insufficient information
- So, need to **absorb** δ digits
- **Question:** Find maximum δ such that cannot emit more than ϵ digits
- **Answer:**

$$\delta = \epsilon + \left\lfloor \log_2 \left(\frac{|\mathrm{det}\,(M)|}{\mathrm{max}\,(|a+b|\,, |c+d|)^2} \right) \right\rfloor$$

- **Fast algorithm** for this involves **basic arithmetic operations** and **counting bits**.

Tensor Information Flow Analysis

- **Arbitrary tensor** in expression tree



$$T = \begin{pmatrix} a & c & e & g \\ b & d & f & h \end{pmatrix}$$

- Want to **emit** ϵ digits
- However insufficient information
- So, need to **absorb** δ_{L} and δ_{R} digits
- **2 Questions:** Find maximum $\delta_{\mathrm{L/R}}$ such that cannot emit more than ϵ digits **regardless of the number of absorptions on the right/left**
- **2 Answers:**Similar to matrix formula

Pi

Ramanujan's amazing formula for π

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} (-1)^n \frac{12(6n)!}{(n!)^3(3n)!} \frac{545140134n + 13591409}{(640320^3)^{n+\frac{1}{2}}}$$

with some magic can be converted into

$$\frac{\sqrt{10005}}{\pi} = \begin{pmatrix} \frac{6795705}{213440} & \frac{6795704}{213440} \end{pmatrix} \prod_{n=1}^{\infty} M_n$$

where

$$\begin{aligned} M_n &= \begin{pmatrix} a_n - b_n - c_n & a_n + b_n - c_n \\ a_n + b_n + c_n & a_n - b_n + c_n \end{pmatrix} \\ a_n &= 10939058860032000n^4 \\ b_n &= (2n-1)(6n-5)(6n-1)(n+1) \\ c_n &= (2n-1)(6n-5)(6n-1)(545140134n + 13591409) \end{aligned}$$

World Record

$51,539,600,000$ decimal digits